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POSSIBLE HEAD-TAIL OSCILLATION OF BUNCHES DUE TO A TRANSVERSE FEEDBACK KICKER

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In high-intensity storage rings, such as KEKB or PEP-II, strong coupled-bunch instabilities due to the high beam-current are expected. Transverse feedback systems, which will be installed to cure the instabilities, should be operated at quite high frequencies corresponding to the bunch-frequencies in these machines. This high-frequency operation can cause an unintended head-tail motion of a bunch due to the slight difference between the kicks at its head and the tail. The nature of this head-tail motions is investigated with a two-particle model and, in addition, its interference with the short-range wake force is studied. It is found that a head-tail motion will be excited even when only the effects of the feedback kicker are taken into account. But in this case, the growth rate is not very large. When we consider the case where the kicker force interferes with the short-range wake force, the growth of the head-tail motion will be enhanced. The results of the analysis are applied to the case of KEKB.

Keywords: The head-tail motion, bunch feedback

1 INTRODUCTION

In large factory-machines, such as KEKB[1] or PEP-II,[2] a large number of bunches will be stored and strong coupled-bunch instabilities are expected. The number of possible modes of instabilities is, in principle, same as the number of bunches itself. Among them, a number of modes whose frequencies coincide with those of some impedance sources start to grow. This causes large-amplitude coherent bunch oscillations or, in the worst case, beam loss. In order to suppress these instabilities, the installation of a feedback system is planned.

In general, the bandwidth of a feedback system must be, at least, half of the bunch frequency, if one wants to damp all possible modes of the instabilities. For example, in the case of KEKB, the bunch frequency is ~500 MHz, and the required bandwidth of the feedback system then amounts to 250 MHz. This means that we must use at least 250 MHz-signals even though the lowest band is used. This frequency is of the same order of that of an rf accelerating system. Also, the feedback kicker, which is one of the constituents of a feedback system, must be operated with this high frequency. We should notice that such feedback systems are quantitatively quite different from the traditional ones in the sense that the feedback kicker is not a low-frequency, but a very high-frequency device that can excite internal bunch oscillations.

The aim of this paper is to investigate possible head-tail oscillations of a bunch due to the transverse feedback kicker. In the first part of this paper we will study the growth of the head-tail motion due to the kicker only. In the later part we investigate the interference of the kicker force with the short-range wake force.

The equations of motion will be derived supposing that our feedback system is of the bunch-by-bunch scheme but not of the mode-by-mode scheme. However, this is only an example and the results here are also true for a mode-by-mode feedback system.

2 BEAM-SIZE GROWTH DUE TO THE TRANSVERSE KICKER

2.1 Basic Function of a Transverse Feedback System

Here, first of all, we clarify the function of a transverse feedback system. The transverse displacement of a bunch from the closed orbit is observed at a certain point in a ring and the displacement-signal thus obtained is sent to a feedback kicker after being delayed and amplified. The delay corresponds to a rotation by 90 degrees in betatron phase. When the bunch passes through the kicker, it receives a transverse momentum kick Δp_t from the kicker. In mathematical terms, this function of a feedback system is expressed by the equation,

$$\frac{\Delta p_t}{p_0} = \Delta x' = i\zeta x, \tag{1}$$

where p_0 is the (central) beam momentum and x is the displacement observed. On the right-hand side, the factor i means the phase rotation by 90 degrees, and the coefficient ζ is a parameter related to the feedback gain. We should note that the usual feedback systems can influence only the center of mass motion of a bunch, since they can not get any information on its internal structure.

2.2 Signal from the Bunches

Now let us imagine M bunches in a ring executing coupled betatron motions with the mode-identifying integer, μ. The integer, μ, ranges in from $-[M/2] + 1$ to

$[M/2]$, where [ ] means the integer part. The motion of a bunch identified by an integer ℓ and is expressed by

$$x_\ell(t) = Ae^{-i(\nu\omega_0 t + \delta_\ell)},$$

where ν is the betatron tune, ω_0 the angular revolution frequency and A is a constant which is common to all the bunches. Here the phase offset δ_ℓ is determined such that the phase difference between the oscillations of adjacent bunches is $\frac{2\pi}{M}\mu$ when it is observed at certain point in the ring. As is well known, the signals from these bunches contain the frequency components,

$$\omega \sim (\bar{\nu} + (Mn + \mu))\omega_0, \qquad (n : \text{ integer})$$

where $\bar{\nu}$ is the difference between ν and the nearest integer. Our feedback system will use, for example, the components corresponding to $n = 0$. The position-information thus obtained is expressed by

$$x(t) = Ae^{-i(\bar{\nu}+\mu)\omega_0 t}.$$

Remembering the range of the integer, μ, we understand that the required bandwidth of a feedback system is a half of the bunch frequency.

2.3 The Kicker Force as a Function of Time

According to the Equation (1), the momentum kick should be obtained by multiplying the observed position by the factor, $i\zeta$:

$$\Delta p_t(t) = i\zeta p_0 x(t) \equiv i\zeta p_0 Ae^{-i(\bar{\nu}+\mu)\omega_0 t} \equiv i\zeta p_0 Ae^{-i\omega_k t}.$$

As the last expression shows, the kicker works at the frequency $\omega_k = (\bar{\nu}+\mu)\omega_0$ (the kicker frequency). We should notice that $\Delta p_t(t)$ is a rapidly-changing function of time, when the mode number, μ, is rather large. This means that a small time slip can cause a non-negligible error of the kick. A particle in a bunch is always executing a longitudinal (synchrotron) oscillation about the center of the bunch even in the case that no coherent oscillation is observed. Consequently particles which form a bunch will receive different kicks from the feedback system as they are wandering to various positions within a bunch. For a particle behind the bunch-center by δt, the kick is expressed by

$$\Delta p_t(t + \delta t) = i\zeta p_0 Ae^{-i\omega_k(t+\delta t)} \simeq i\zeta p_0 Ae^{-i(\bar{\nu}+\mu)\omega_0 t} + \zeta p_0 \delta t \omega_k Ae^{-i(\bar{\nu}+\mu)\omega_0 t}.$$

Now we concentrate our attention to a specific bunch whose phase offset, δ_ℓ, is 0. Its motion is expressed by

$$x_0(t) = Ae^{-i\omega_0 \nu t}.$$

The center of this bunch passes through the kicker at the moments,

$$t_m = m(2\pi/\omega_0) = mT_0, \qquad (m = \cdots, -1, 0, 1, 2, \cdots)$$

here T_0 is the revolution period. The kick for this bunch is

$$\begin{aligned}\Delta p_t &= i\zeta p_0 A e^{-i(\bar{\nu}+\mu)\omega_0 m T_0} + p_0 \zeta \omega_k \delta t A e^{-i(\bar{\nu}+\mu)\omega_0 m T_0} \\ &= i\zeta p_0 A e^{-i\nu\omega_0 m T_0} + p_0 \zeta \omega_k \delta t A e^{-i\nu\omega_0 m T_0}.\end{aligned}$$

The momentum kick obtained above is given to the bunch at the moment at which the bunch passes through the kicker (an impulsive force). But we assume that this momentum change is given to it smoothly over one revolution period. Then the kicker force is effectively given by, $\Delta p_t / T_0$, i.e.,

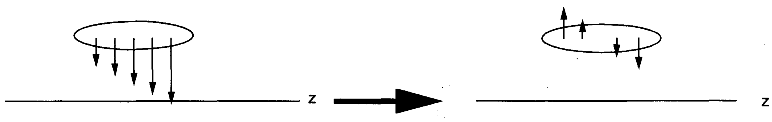
$$\begin{aligned}F_{\text{kick}} &= i\zeta \frac{p_0}{T_0} A e^{-i\nu\omega_0 m T_0} + \frac{p_0}{T_0} \zeta \omega_k \delta t A e^{-i\nu\omega_0 m T_0} \\ &\simeq i\zeta \frac{p_0}{T_0} x_0(t) + \frac{p_0}{T_0} \zeta \omega_k \delta t x_0(t).\end{aligned} \tag{2}$$

The last expression is basic to our investigation, because it gives the relation between the kicker force and the center of mass of a bunch.

2.4 Equations of Motion

In this section, we neglect the wake force, that is, the forces acting on the particles in a bunch are the kicker force derived above and the restoring force due to focusing magnets. Since we know all forces, we can, in principle, trace the motion of all the particles in a bunch. But actually, the number of particles in a bunch, N, is so large that it is not fruitful to treat the motion of all of them. Instead, we apply the two-particle model[4] to the system. In this model, a bunch consists of only two macro-particles each of which is formed by $N/2$ elementary particles. Even though this model is very simple, we can extract many reliable and important results.

The two equations, which describe the motion of the macro-particles, are obtained by the following procedure. The transverse positions of the macro-particles are



(a) All parts of a bunch are kicked with the same sign but the amplitude of the kick depends on the longitudinal position within the bunch.

(b) Only the head-tail kick component is extracted. The sign of the kick in the head is opposite to that in the tail.

FIGURE 1: (a) The kick force is a function of the position of a particle within a bunch. We can understand that the kick is given by the sum of the main dipole-kick and a correction which is nothing but the head-tail force. (b) In our analysis only the head-tail force is extracted, that is, the main dipole kick is eliminated.

expressed by a pair of complex numbers, x_1 and x_2. The real parts of these numbers represent their positions, while the imaginary parts are closely related to their momenta. Since the bunch consists of only 2 particles, the center of mass of this bunch is simply expressed by $(x_1 + x_2)/2$. Then the kicker force is obtained by replacing $x_0(t)$ with $(x_1 + x_2)/2$ in the expression, (2). The equations of motion for these two macro-particles are, then

$$m_0\gamma\ddot{x}_1 = -\kappa_1 x_1 + i\zeta\frac{p_0}{T_0}\frac{x_1+x_2}{2} + \omega_k\frac{p_0}{T_0}\zeta\delta t_1\frac{x_1+x_2}{2},$$

$$m_0\gamma\ddot{x}_2 = -\kappa_2 x_2 + i\zeta\frac{p_0}{T_0}\frac{x_1+x_2}{2} + \omega_k\frac{p_0}{T_0}\zeta\delta t_2\frac{x_1+x_2}{2},$$

where $m_0\gamma$ is the mass of the particle, the term $-\kappa_i x_i$ the restoring force due to focusing quadrupole magnets and the dot on x represents the differentiation with respect to time, t. Note that the force from the kicker is expressed by the sum of the main dipole-kick and a small correction term. We call this correction term the head-tail force, because this force has the opposite sign in the head and the tail of the bunch. Since this force is a possible source of head-tail oscillations of a bunch, we extract this term from the total kick-force. In other words, we eliminate the force corresponding to the main dipole-kick as shown in Figure 1. In this picture, the equation of motions are reduced to

$$m_0\gamma\ddot{x}_1 = -\kappa_1 x_1 + \omega_k\frac{p_0}{T_0}\zeta\delta t_1\frac{x_1+x_2}{2},$$

$$m_0\gamma\ddot{x}_2 = -\kappa_2 x_2 + \omega_k\frac{p_0}{T_0}\zeta\delta t_2\frac{x_1+x_2}{2}.$$

One may think that it is not correct to extract only the head-tail force omitting the main dipole-kick. This approach is, however, not unrealistic as explained below. The feedback system is switched on only when some impedance source excites the oscillation. Confronting this impedance source, the feedback suppresses the oscillation. At a certain amplitude of the oscillation, the forces from the impedance and from the feedback are balanced, and the amplitude of the oscillation is kept constant. In this equilibrium state, the main dipole kick is effectively canceled out by the kick by the impedance.

In order to take the effects of the longitudinal oscillation into account, we express those of the macro-particles by,

$$\delta t_1 = -\frac{\sigma_z}{c}\sin(ck_s t), \qquad \delta t_2 = \frac{\sigma_z}{c}\sin(ck_s t),$$

where k_s is the wave number of the synchrotron oscillation, σ_z the bunch length and c the speed of light. As the above expressions show, these oscillations are out of phase by 180 degrees. Substituting these expressions into the equations of motion and converting the independent variable from t to s (distance from a reference point on the central trajectory), we obtain the set of equations,

$$x_1'' + k_1^2 x_1 + \frac{1}{2}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)(x_1 + x_2) = 0, \tag{3}$$

$$x_2'' + k_2^2 x_2 - \frac{1}{2}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)(x_1 + x_2) = 0, \tag{4}$$

where C is the circumference of the ring, the primes denote differentiation with respect to s, and $k_{1,2}$ are the betatron wave numbers taking the chromaticity[5] into account. They are expressed by

$$k_1 = k_\beta(1 - \xi\sigma_\epsilon\cos(k_s s)), \qquad k_2 = k_\beta(1 + \xi\sigma_\epsilon\cos(k_s s)),$$

where k_β is the betatron wave number for a synchronous particle and σ_ϵ is the amplitude of the energy oscillation associated with the longitudinal oscillation. The Equations (3) and (4), are the fundamental equations describing our system.

2.5 Solving the Equations

We try to solve the simultaneous Equations (3) and (4), by variation of constants, that is, we assume that the solutions have the forms

$$x_1(s) = X_1(s)e^{-ik_\beta s + i\varphi\sin(k_s s)}, \quad x_2(s) = X_2(s)e^{-ik_\beta s - i\varphi\sin(k_s s)}, \tag{5}$$

where $X_1(s)$ and $X_2(s)$ are amplitudes of oscillation to be determined, and φ is the head-tail phase defined by

$$\varphi := \frac{\xi k_\beta \sigma_z}{\alpha},$$

with the parameter, α the momentum compaction factor. Now we substitute the trial solutions, (5) into the Equations (3) and (4). In this case, we make several approximations based on the fact that φ is much smaller than unity. In addition, we neglect the second derivative of X_i with respect to s. This means that our analysis does not cover extremely rapid motions.

With these approximations we obtain the following equations.

$$X_1'(s) + \frac{i}{4k_\beta}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)\Big\{X_1(s) + X_2(s)(1 - 2i\varphi\sin(k_s s))\Big\} = 0,$$

$$X_2'(s) - \frac{i}{4k_\beta}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)\Big\{X_1(s)(1 + 2i\varphi\sin(k_s s)) + X_2(s)\Big\} = 0.$$

It is convenient to define a new parameter which measures the strength of the effects of the kicker,

$$\mathcal{K} := \frac{1}{4k_\beta}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c} \equiv \frac{1}{4}\beta_k\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}, \tag{6}$$

where β_k is the beta function at the kicker. This parameter has the dimension of L^{-1}, and is proportional not only to the required damping rate of the feedback system but also to the kicker frequency.

We can express the equation in a matrix form:

$$\begin{bmatrix} X_1'(s) \\ X_2'(s) \end{bmatrix} = i\mathcal{K}\sin(k_s s)\begin{bmatrix} -1 & -(1 - 2i\varphi\sin(k_s s)) \\ 1 + 2i\varphi\sin(k_s s) & 1 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}.$$

The matrix appeared here is not symmetric, therefore it is not diagonalizable with any unitary transformation.[3] This is a consequence of the fact that we are not describing an isolated two-body system, but a system driven by an external force which is not symmetric under the exchange of these particles. Then we are limited to transforming the matrix into a triangular one.

The transformation matrix is found to be

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \tag{7}$$

and the new variables are

$$X_-(s) := \frac{X_1(s) - X_2(s)}{\sqrt{2}}, \qquad X_+(s) := -\frac{X_1(s) + X_2(s)}{\sqrt{2}},$$

They represent the relative position and the center of mass of the 2 macro-particles. The transformed equation of motion is

$$\begin{bmatrix} X'_-(s) \\ X'_+(s) \end{bmatrix} = i\mathcal{K}\sin(k_s s)\begin{bmatrix} -2i\varphi\sin(k_s s) & 2 \\ 0 & 2i\varphi\sin(k_s s) \end{bmatrix}\begin{bmatrix} X_-(s) \\ X_+(s) \end{bmatrix}. \tag{8}$$

It is easy to solve the equation for $X_+(s)$. Its explicit form is

$$X'_+(s) = -2\mathcal{K}\varphi\sin^2(k_s s)X_+(s).$$

and the solution is

$$X_+(s) = X_+(0)e^{-\mathcal{K}\varphi\left(s - \frac{1}{2k_s}\sin(2k_s s)\right)}, \tag{9}$$

where $X_+(0)$ is the initial value of $X_+(s)$. The last expression shows that the motion of the center of mass is damped although the main dipole term of the kicker has been eliminated. However, the damping rate is very small compared to that due to the main dipole-kick term (see Section 2.6).

The next job is to find the solution for $X_-(s)$. The equation is

$$X'_-(s) = 2\mathcal{K}\varphi\sin^2(k_s s)X_-(s) + 2i\mathcal{K}\sin(k_s s)X_+(s) \tag{10}$$

from (8). We suppose that the solution to this equation is approximately given by the sum of two terms:

$$X_-(s) = \bar{X}_-(s) + \tilde{X}_-(s),$$

where $\bar{X}_-(s)$ is the solution to the equation which is obtained by neglecting the driving (inhomogeneous) term, while $\tilde{X}_-(s)$ is a special solution which would be obtained when only the driving term survives of the right-hand side of (10). The initial value of $\tilde{X}_-(s)$ is 0 and consequently the initial value of $\bar{X}_-(s)$ is that of $X_-(s)$ itself. In the words of physics, $X_-(s)$ is a superposition of the free oscillation and a forced (driven) oscillation which begins to grow at $s = 0$.

The equation for $\bar{X}_-(s)$ is same as that for $X_+(s)$ except for the sign on the right-hand side. Then the solution is easily found to be

$$\bar{X}_-(s) = X_-(0)e^{\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)}. \tag{11}$$

This solution tells us that the free oscillation part of $X_-(s)$ is a diverging function of time. Next we must find the special solution which satisfies the relation,

$$X'_-(s) = 2i\mathcal{K}\sin(k_s s)X_+(s).$$

We try the following candidate for the solution:

$$p\cos(k_s s)X_+(0)e^{-\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)} + \text{ a constant term}$$

with the constant p to be determined. Substituting the candidate into the equation, we find

$$\begin{aligned}
&- pk_s\sin(k_s s)X_+(0)e^{-\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)}\\
&\qquad + p\cos(k_s s)\left(-\mathcal{K}(1-\cos(k_s s)\right)X_+(0)e^{-\mathcal{K}\varphi\left(s-\frac{1}{k_s}\sin(k_s s)\right)}\\
&\qquad\qquad = 2i\mathcal{K}\sin(k_s s)X_+(0)e^{-\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)}.
\end{aligned}$$

Now we restrict ourselves to the cases of $\mathcal{K} \ll k_s$. We will discuss the validity of the restriction later. Under this condition the second term of the left-hand side can be neglected. Then the constant p is easily determined and the solution is

$$\tilde{X}_-(s) = -\frac{2i\mathcal{K}}{k_s}X_+(0)\left(\cos(k_s s)e^{-\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)} - 1\right). \tag{12}$$

We combine the last solution and (11) to obtain the solution for $X_-(s)$:

$$\begin{aligned}
&X_-(s) =\\
&X_-(0)e^{\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)} - \frac{2i\mathcal{K}}{k_s}X_+(0)\left(\cos(k_s s)e^{-\mathcal{K}\varphi\left(s-\frac{1}{2k_s}\sin(2k_s s)\right)} - 1\right).
\end{aligned}$$

From the last expression, we find that the motion is a superposition of a diverging part and a damping part plus a constant term. The diverging term dominates under our restriction even at $s \approx 0$ if $X_-(0)$ and $X_+(0)$ are in the same order. The behavior of $x_1(s)$ and $x_2(s)$ is easily derived from the results by applying the inverse matrix of Equation (7). Both coordinates diverge while keeping their center-of-mass very close to the center trajectory.

From these discussions, we have reached important results:

1. According to the expression, (9), the center of mass will be damped (if $\varphi > 0$ as usual) even though the main dipole kick is eliminated from the equations of motion. The damping per unit time is $\mathcal{K}c\varphi$. The fact that the damping rate is proportional to φ means that there is no damping if the chromaticity vanishes.
2. The relative coordinate, i.e., the transverse distance of the two particles, will diverge. The growth per unit time is $\mathcal{K}c\varphi$. Like the center-of-mass behavior, there is no amplitude variation if the chromaticity is 0.

2.6 Numerical Consideration

In the last section, we found that the transverse distance of x_1 and x_2 diverges with a growth rate of $\mathcal{K}c\varphi$ [s^{-1}]. Here we give a numerical estimate on this growth rate for KEKB Low-Energy Ring. The KEKB main parameters concerning the feedback systems are as follows:

TABLE 1: The KEKB main parameters concerning the transverse feedback systems.

	LER		*HER*
energy (GeV)	3.5		8.0
circumference (m)		3016	
harmonic number		5120	
bunch frequency (MHz)		509	
betatron tune		~45	
average beta (m)		~10	
bunch length (mm)		3~5	
synchrotron tune		0.01~0.02	
transverse radiation damping time (ms)	86		46
momentum compaction factor		1×10^{-4}	
# of particles/bunch	3.3×10^{10}		1.4×10^{10}

At first, we calculate the kicker parameter, $\mathcal{K}$,

$$\mathcal{K} = \frac{\omega_k \zeta \sigma_z}{4\omega_\beta C}.$$

Here the damping coefficient, ζ, is given by

$$\zeta = 2g_d \frac{1}{\sqrt{\beta_m \beta_k}},$$

where g_d is the damping rate of the feedback system in turn^{-1} and β_k, β_m are the beta functions at the feedback kicker and the monitor, respectively. Assuming the value 1/100 turn^{-1} as a typical damping rate and using the average value of β, i.e. about 10m, for β_k and β_m, ζ is found to be $2 \times 10^{-3}[\mathrm{m}^{-1}]$. For the kicker frequency, we use the highest possible value in KEKB, $2\pi \times 254$ MHz ($f_{\mathrm{bunch}}/2$), which corresponds to the mode number of the coupled-bunch instability 2560. Using these values, we obtain an approximate value of $\mathcal{K}$:

$$\mathcal{K} \approx 5 \times 10^{-8}\mathrm{m}^{-1}. \tag{13}$$

When analyzing the head-tail phenomena, we can roughly estimate the effect of a perturbation, e.g. the wake force, by comparing the frequency due to the perturbation with the synchrotron frequency. In our case we should compare the above value of the kicker parameter with the wavelength of the synchrotron oscillation. According to Table 1, the synchrotron tune will be 0.01~0.02. This corresponds to

$$k_s = (2 \sim 4) \times 10^{-5}\mathrm{m}^{-1}. \tag{14}$$

Our assumption, $\mathcal{K} \ll k_s$, which we made when deriving (12), was reasonable as far as we adapt the KEKB parameters.

For obtaining the growth rate of the head-tail oscillation due to the kicker, we must calculate the head-tail phase. We assume the chromaticity, ξ, is about $1/\nu_\beta \approx 1/45$, then φ is obtained to be 0.1. The growth rate, g_r, is given by

$$g_r = \mathcal{K}\varphi c \approx 1.6\mathrm{s}^{-1}.$$

This is much smaller than the damping rate due to the radiation, and hence we need not be concerned with this phenomenon.

3 INTERFERENCE WITH THE WAKE FORCE

3.1 Equations of Motion and their Solutions

In this section we will incorporate the effects of the short-range wake field. The transverse wake force will introduce, like the head-tail force of the transverse kicker, the differences in the transverse momentum-change depending on the relative longitudinal position of particles. The two forces, the kicker's head-tail force and the wake force, can interfere with each other and some different phenomena, which could not occur when they affect on the system independently, can occur.

Here we analyze the system with the 2-particle model again. The equations of motion are obtained by adding a wake-origin term to the Equation (3) or (4). This is a consequence of the fact that the trailing particle is kicked by the wake which is generated by the leading particle. As we discussed, the two particles are executing the longitudinal oscillation and, consequently, their relative longitudinal position is altered every half of the synchrotron-oscillation period. The equations in the first half period are

$$x_1'' + k_-^2 x_1 + \frac{1}{2}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)(x_1 + x_2) = 0,$$

$$x_2'' + k_+^2 x_2 - \frac{1}{2}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)(x_1 + x_2) - \frac{(N/2)e}{(E/e)C}W_0 x_1 = 0,$$

where E/e is the beam energy in volts and W_0 is the transverse wake function. W_0 is a function of the position of the particle 1 relative to the particle 2, but here, we regard it as a positive constant (of course real) over the length of the bunch ("constant wake approximation").

Again we assume the form of the solutions:

$$x_1(s) = X_1(s)e^{-ik_\beta s + i\varphi\sin(k_s s)}, \qquad x_2(s) = X_2(s)e^{-ik_\beta s - i\varphi\sin(k_s s)},$$

and the equations of motion for $X_1(s)$ and $X_2(s)$ are

$$X_1'(s) + \frac{i}{4k_\beta}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)\Big\{X_1(s) + X_2(s)(1 - 2i\varphi\sin(k_s s))\Big\} = 0,$$

$$X_2'(s) - \frac{i}{4k_\beta}\frac{\zeta\omega_k}{C}\frac{\sigma_z}{c}\sin(k_s s)\Big\{X_1(s)(1 + 2i\varphi\sin(k_s s)) + X_2(s)\Big\}$$
$$- i\frac{1}{2k_\beta}\frac{(N/2)eW_0}{(E/e)C}(1 + 2i\varphi\sin(k_s s))X_1(s) = 0.$$

As in the previous case, let us express the equations in a matrix form:

$$\begin{bmatrix} X_1'(s) \\ X_2'(s) \end{bmatrix}$$
$$= i\begin{bmatrix} -\mathcal{K}\sin(k_s s) & -\mathcal{K}\sin(k_s s)(1 - 2i\varphi\sin(k_s s)) \\ (1 + 2i\varphi\sin(k_s s))(\mathcal{K}\sin(k_s s) + \mathcal{W}) & \mathcal{K}\sin(k_s s) \end{bmatrix}$$
$$\times \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix},$$

where the parameter, $\mathcal{W}$, is given by

$$\mathcal{W} := \frac{1}{2k_\beta} \frac{(N/2)eW_0}{(E/e)C}. \tag{15}$$

This parameter, like the parameter $\mathcal{K}$, has the dimension L^{-1} and is proportional to the bunch current and the wake strength W_0. Following the previous procedure, we try to solve the equation by transforming the matrix into a triangular one. The secular equation for the above matrix is

$$\lambda^2 + \mathcal{K}\mathcal{W}\sin(k_s s) + 4\mathcal{K}\mathcal{W}\varphi^2 \sin^3(k_s s) + 4\varphi^2\mathcal{K}^2 \sin^4(k_s s) = 0.$$

Since the factor, $\mathcal{W}\mathcal{K}\sin(k_s s)$, is positive in the first half of the synchrotron period, the eigenvalues become

$$\lambda_\pm = \pm i\sqrt{\mathcal{K}\mathcal{W}}\sqrt{\sin(k_s s)}\sqrt{1 + 4\varphi^2 \sin^2(k_s s) + 4\varphi^2(\mathcal{K}/\mathcal{W})\sin^3(k_s s)}.$$

We can transform the matrix into a triangular one by a unitary matrix which is defined in $0 < s < \pi/k_s$, given by

$$T' = \frac{1}{\sqrt{2S + \mathcal{W}/\mathcal{K}}} \begin{bmatrix} \sqrt{S} & \frac{-\mathcal{K}\sqrt{S} + i\sqrt{\mathcal{W}\mathcal{K}}Q}{\mathcal{K}(1+2i\varphi S)} \\ \frac{-\mathcal{K}\sqrt{S} - i\sqrt{\mathcal{W}\mathcal{K}}Q}{\mathcal{K}(1-2i\varphi S)} & -\sqrt{S} \end{bmatrix}. \tag{16}$$

Here we introduced the new symbols,

$$S := \sin(k_s s), \qquad Q := \sqrt{1 + 4\varphi^2 \sin^2(k_s s) + 4\varphi^2(\mathcal{K}/\mathcal{W})\sin^3(k_s s)}$$

to simplify the notation. This matrix is a little more complicated than (7) and depends on time. But it is easy to verify that this is reduced to (7) for a very small chromatic phase shift φ, if the wake force is much weaker than the kicker force. To the contrary, if the wake force is very strong, the matrix is roughly given by

$$T' \simeq \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{17}$$

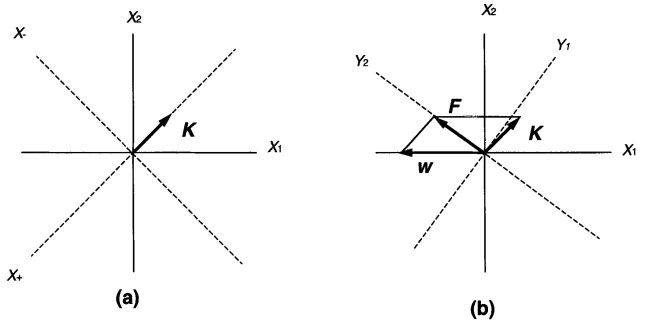


FIGURE 2: (a) The orthogonal transformation expressed by (6). (b) An intuitive image of the unitary transformation, (16). Under the new coordinate system the sum of kicker force and wake force has only the Y_1 component.

In particular, every half period of the synchrotron-oscillation, the matrix exactly coincides with (17). The meaning of the matrix is that the roles of the particles 1 and 2 should be exchanged after a phase shift by ±90 degrees.

Here we try to get an intuitive understanding of the unitary transformation, given by Equation (16), as well as of the orthogonal transformation, Equation (7). At first we assume that only the kicker force exists. This force tries to move the center of mass but does not have any effect on the relative coordinate, $x_1 - x_2$. The vector **K** in Figure 2(a) represents this force. The amplitude and the sign of the force change as time passes but its direction stays the same. The orthogonal transformation, (7), introduces a coordinate transformation, from the old x_1-x_2 system to the new one, the x_--x_+ system. Under this new coordinate system, the force has only the x_- component.

Next we study the case where the wake force is taken into account. The force given to the system is the sum of the kicker force and the wake force. The kicker force is the same as the first case. If the particle 1 is followed by the particle 2, the wake force is parallel with the coordinate, x_1. Then the sum will be the vector **F** in Figure 2(b). The unitary transformation, (16), corresponds to the coordinate transformation from the original one to a new one where the force lies only on one coordinate, namely, Y_1 direction. The other coordinate, which is perpendicular to Y_1, is called Y_2. The amplitude of the kicker force is dynamically changing and, therefore, the direction and the amplitude of the vector-sum of the forces is changing as a function of time.

After making the matrix triangular, the equation becomes

$$\begin{bmatrix} Y_1'(s) \\ Y_2'(s) \end{bmatrix} = i \begin{bmatrix} i\sqrt{\mathcal{WK}}\sqrt{S}Q & \frac{(2\mathcal{K}S-\mathcal{W})-4\varphi^2\mathcal{W}S^2-2iQ\sqrt{\mathcal{WK}}\sqrt{S}}{1+2i\varphi S} \\ 0 & -i\sqrt{\mathcal{WK}}\sqrt{S}Q \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}.$$

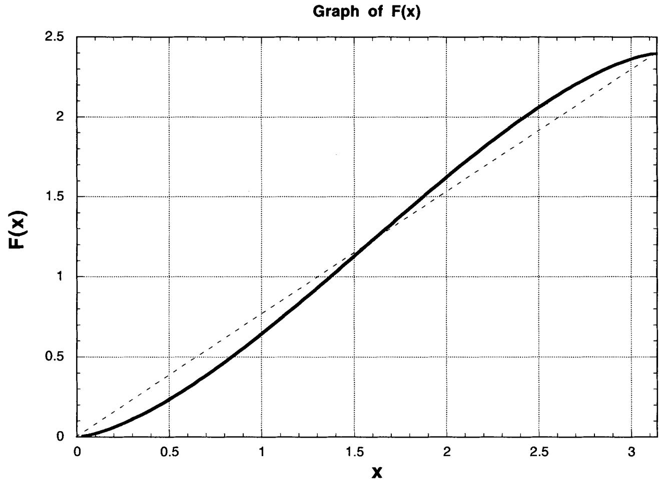


FIGURE 3: A graph of the function, $F(x)$. The curve is close to a linear function.

We can safely neglect the second order of φ, and the above equation is simplified to

$$\begin{bmatrix} Y_1'(s) \\ Y_2'(s) \end{bmatrix} = \tag{18}$$
$$\begin{bmatrix} -\sqrt{\mathcal{WK}}\sqrt{S} & \left((2i\mathcal{K}S - i\mathcal{W}) + 2\sqrt{\mathcal{WK}}\sqrt{S}\right)(1 - 2i\varphi S) \\ 0 & \sqrt{\mathcal{WK}}\sqrt{S} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}.$$

We find that the factors $\pm\sqrt{\mathcal{WK}}\sqrt{\sin(k_s s)}$ appear as the diagonal components. It is important to compare them with those obtained when we analyzed the effects of the kicker only. In that case, the diagonal positions were occupied by $\pm 2\mathcal{K}\varphi\sin^2(k_s s)$ as shown in (8). A rough comparison shows that the parameter, $2\mathcal{K}\varphi$ is now replaced with the interference factor, $\sqrt{\mathcal{WK}}$. This is a result of the mixture of $\mathcal{W}$-$\mathcal{K}$shown in Figure 2(b).

The equation for $Y_2(s)$ has no inhomogeneous term and is solved easily. The solution is

$$Y_2(s) = Y_2(0)e^{\sqrt{\mathcal{WK}}\int_0^s \sqrt{\sin(k_s s')}ds'}.$$

Defining the function $F(x) = \int_0^x \sqrt{\sin(u)}du$, $Y_2(s)$ is expressed by

$$Y_2(s) = Y_2(0)e^{\frac{\sqrt{\mathcal{WK}}}{k_s}F(k_s s)}.$$

A graph of the function $F(x)$ is given in Figure 3. This function has the value 0 at $x = 0$ and ~ 2.4 at $x = \pi$. Looking at this graph, we find that this function can be approximated with a linear function, as, where a is given by $a = \int_0^\pi \sqrt{\sin t}\, dt/\pi \approx 2.4/\pi$. With this approximation, $Y_2(s)$ is expressed by

$$Y_2(s) = Y_2(0)e^{a\sqrt{\mathcal{WK}}s}. \tag{19}$$

This expression shows us that $Y_2(s)$ converges with time. The damping rate is $ac\sqrt{\mathcal{WK}}$.

Next we should solve the equation for $Y_1(s)$:

$$\begin{aligned} Y_1'(s) = & -\sqrt{\mathcal{WK}}\sqrt{\sin(k_s s)}Y_1(s) \\ & + (1 - 2i\varphi \sin(k_s s))\left((2i\mathcal{K}\sin(k_s s) - i\mathcal{W}) + 2\sqrt{\mathcal{WK}}\sqrt{\sin(k_s s)}\right)Y_2(s). \end{aligned}$$

derived from Equation (18). Here we make the same approximations as for the solution of Equation (10) for $X_-(s)$. The solution is the sum of a free oscillation and the driven oscillation:

$$Y_1(s) = \bar{Y}_1(s) + \tilde{Y}_1(s),$$

where $\bar{Y}_1(s)$ is the solution to the homogeneous equation obtained by neglecting the inhomogeneous term. On the other hand, $\tilde{Y}_1(s)$ is the forced-oscillation solution. Again the initial value of $\tilde{Y}_1(s)$ should be 0.

The equation for $\bar{Y}_1(s)$ is simple and the solution is obtained to be

$$\bar{Y}_1(s) = Y_1(0)e^{-a\sqrt{\mathcal{WK}}s}. \tag{20}$$

This is just same as the solution for $Y_2(s)$ except for the sign in the exponential factor.

In order to solve the equation for the forced oscillation, $\tilde{Y}_1(s)$, we make an approximation that over the range of $0 < s < \pi/k_s$, $\sqrt{\sin(k_s s)}$ is replaced by the constant, a, the same one as introduced when we approximate the function $F(x)$. It is quite reasonable, because $\frac{d(ax)}{dx} \approx \frac{dF}{dx} = \sqrt{\sin x}$. Additionally, we use the expression (19) for $Y_2(s)$ in Equation (18). After these preparations we have an equation for $Y_1(s)$:

$$\tilde{Y}_1'(s) = (1 - 2i\varphi \sin(k_s s)) \left[(2i\mathcal{K} \sin(k_s s) - i\mathcal{W}) + 2a\sqrt{\mathcal{W}\mathcal{K}}\right] Y_2(0) e^{a\sqrt{\mathcal{W}\mathcal{K}}s}.$$

We examine the following trial solution:

$$\begin{aligned}\tilde{Y}_1(s) = \Big[& p_1 \left(e^{(ik_s + a\sqrt{\mathcal{W}\mathcal{K}})s} - 1\right) + p_2 \left(e^{(-ik_s + a\sqrt{\mathcal{W}\mathcal{K}})s} - 1\right) \\ & + p_3 \left(e^{(2ik_s + a\sqrt{\mathcal{W}\mathcal{K}})s} - 1\right) + p_4 \left(e^{(-2ik_s + a\sqrt{\mathcal{W}\mathcal{K}})s} - 1\right) \\ & + p_5 \left(e^{a\sqrt{\mathcal{W}\mathcal{K}}s} - 1\right)\Big] Y_2(0),\end{aligned}$$

where p_i $(i = 1, \cdots, 5)$ are constants to be determined. The term, -1, is inserted into each parentheses to ensure that $\tilde{Y}_1(s)$ has the value of 0 at $s = 0$. Substituting the trial solution into the expression, we determine the five coefficients. As a result we get,

$$\begin{aligned}\tilde{Y}_1(s) = \Bigg[& \frac{e^{a\sqrt{\mathcal{K}\mathcal{W}}s}}{k_s^2 + 4a^2\mathcal{W}\mathcal{K}} \Big\{-2k_s(iK - \mathcal{W}\varphi - 2i\varphi a\sqrt{\mathcal{K}\mathcal{W}})(\cos(k_s s) - 1) \\ & + 2(iKa\sqrt{\mathcal{K}\mathcal{W}} - a\varphi\mathcal{W}\sqrt{\mathcal{K}\mathcal{W}} - 2ia^2\mathcal{W}\mathcal{K}) \sin(k_s s)\Big\} \\ & + \frac{2\mathcal{K}\varphi e^{a\sqrt{\mathcal{K}\mathcal{W}}s}}{4k_s^2 + a^2\mathcal{W}\mathcal{K}} \left(-2k_s \sin(2k_s s) - a\sqrt{\mathcal{W}\mathcal{K}}(\cos(2k_s s) - 1)\right) \\ & + \left(2 - i\frac{1}{a}\sqrt{\mathcal{W}/\mathcal{K}} + 2\frac{1}{a}\sqrt{\mathcal{K}/\mathcal{W}}\right)\left(e^{a\sqrt{\mathcal{K}\mathcal{W}}s} - 1\right)\Bigg] Y_2(0). \qquad (21)\end{aligned}$$

This solution consists of a constant term and diverging terms which grow with the rate of $a\sqrt{\mathcal{K}\mathcal{W}}c$. Combining the last expression with the free-oscillation solution, (20), we will be able to obtain the general solution for $Y_1(s)$.

3.2 Analysis with a Matrix

In the last section, we have obtained solutions both for $Y_1(s)$ and $Y_2(s)$, which are equivalent to those of $X_1(s)$ and $X_2(s)$. Here we check the stability of a bunch with the method which is usually applied when discussing the nature of the head-tail instability.

After one half of a synchrotron period, i.e., at $s = \frac{\pi}{k_s}$, $X_1(s)$ and $X_2(s)$ have values,

$$X_1\left(\frac{\pi}{k_s}\right) = X_1(0)e^g,$$

$$X_2\left(\frac{\pi}{k_s}\right) = X_1(0)\Bigg[i\left(\frac{4k_s\mathcal{K} - 8\varphi a\sqrt{\mathcal{W}\mathcal{K}}}{k_s^2 + 4a^2\mathcal{W}\mathcal{K}}e^g - \frac{1}{a}\sqrt{\mathcal{W}/\mathcal{K}}\left(e^g - 1\right)\right)$$
$$+ \left(\frac{-4\mathcal{W}\varphi k_s}{k_s^2 + a^2\mathcal{W}\mathcal{K}}e^g + \left(2 + 2\frac{1}{a}\sqrt{\mathcal{K}/\mathcal{W}}\right)\left(e^g - 1\right)\right)\Bigg] + X_2(0)e^{-g},$$

where g is given by

$$g := \frac{\sqrt{\mathcal{W}\mathcal{K}}}{k_s}F(\pi). \tag{22}$$

Among the terms in (21), the $\sin(k_s s)$ terms and those with 2 times of the synchrotron-oscillation frequency vanish at $s = n\pi/k_s$. We introduce a new parameter, Υ, which is defined by

$$\Upsilon := \left[\frac{4k_s\mathcal{K} - 8\varphi a\sqrt{\mathcal{W}\mathcal{K}}}{k_s^2 + 4a^2\mathcal{W}\mathcal{K}}e^g - \frac{1}{a}\sqrt{\mathcal{W}/\mathcal{K}}\left(e^g - 1\right)\right]$$
$$+ i\left[\frac{4\mathcal{W}\varphi k_s}{k_s^2 + a^2\mathcal{W}\mathcal{K}}e^g - \left(2 + 2\frac{1}{a}\sqrt{\mathcal{K}/\mathcal{W}}\right)\left(e^g - 1\right)\right]. \tag{23}$$

The relation between the initial values and those at $s = \frac{2\pi}{k_s}$ can be expressed in the matrix form:

$$\begin{bmatrix} X_1(\pi/k_s) \\ X_2(\pi/k_s) \end{bmatrix} = \begin{bmatrix} e^g & 0 \\ i\Upsilon & e^{-g} \end{bmatrix}\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}. \tag{24}$$

In the latter half of the synchrotron period, the equations of motion are exactly the same except that the roles of particles 1 and 2 are exchanged. The matrix describing the motion is obtained by exchanging the components row by row and column by column in (24). Then the transfer matrix for one synchrotron period is

$$\begin{bmatrix} e^g & 0 \\ i\Upsilon & e^{-g} \end{bmatrix}\begin{bmatrix} e^{-g} & i\Upsilon \\ 0 & e^g \end{bmatrix} = \begin{bmatrix} 1 & i\Upsilon e^g \\ i\Upsilon e^{-g} & -\Upsilon^2 + 1 \end{bmatrix}. \tag{25}$$

The stability of the beam is governed by the eigenvalues of this matrix, $\lambda_\pm$,

$$\lambda_\pm = e^{\pm i\rho},$$

where the relation between Υ and a newly introduced parameter, ρ, is given by $\sin(\rho/2) = \Upsilon/2$. The eigen-vector corresponding λ_+ is the in-phase (i.e. coherent) oscillation of two particles, while the one corresponding to λ_- is the 180-degrees out-of-phase oscillation in the limit of small beam current.[4]

Since both the real and imaginary parts of Υ are not zero, also ρ has finite real and imaginary parts. The finite imaginary part means that the absolute value of $\lambda_\pm$ is not 1, i.e. the motion is growing or damping. The growth (if negative, damping) rate is given by

$$g_r = \mp\frac{\Im(\rho)}{T_s},$$

where $T_s = 2\pi/ck_s$ is the synchrotron oscillation period. When $|\Upsilon| \ll 1$, ρ is almost equal to Υ and the growth rate is proportional to the imaginary part of Υ. In our case, the growth rate is given by

$$g_r = \frac{ck_s}{2\pi}\left[\frac{4\mathcal{W}\varphi k_s}{k_s^2 + a^2\mathcal{W}\mathcal{K}}e^g - \left(2 + 2\frac{1}{a}\sqrt{\mathcal{K}/\mathcal{W}}\right)\left(e^g - 1\right)\right].$$

Here we chose only the growth rate of the eigenstate of λ_- mode, because it is the growing one when the chromaticity and the momentum compaction factor are positive.

It is useful to compare these results with those of an analysis of the head-tail instability with the two-particle model.[4] In this case, a similar but less complex parameter, $\Upsilon_{\text{h-t}}$ is used, i.e.,

$$\Upsilon_{\text{h-t}} = \mathcal{W}\frac{\pi}{k_s} + i\frac{4\varphi\mathcal{W}}{k_s}. \tag{26}$$

The matrix for one synchrotron-oscillation period is simply obtained by replacing Υ with $\Upsilon_{\text{h-t}}$ in (25). The imaginary part, $i\frac{4\varphi\mathcal{W}}{k_s}$, gives the growth rate,

$$g_r' = \frac{2}{\pi}\varphi c\mathcal{W}.$$

We easily find that the first term of the imaginary part of (23) corresponds to this growth term. It can be rewritten as follows:

$$\frac{4\mathcal{W}\varphi k_s e^g}{k_s^2 + a^2\mathcal{W}\mathcal{K}} = \frac{4\varphi\mathcal{W}}{k_s}\frac{e^g}{1 + \frac{a^2\mathcal{W}\mathcal{K}}{k_s^2}} = \frac{4\varphi\mathcal{W}}{k_s}\frac{e^g}{1 + (g/\pi)^2}.$$

Comparing the last expression with the imaginary part of (26) we reach the result that the strength of the head-tail instability is enhanced by the factor,

$$f_e := \frac{e^g}{1+(g/\pi)^2} = \frac{e^{\sqrt{\mathcal{W}\mathcal{K}}F(\pi)/k_s}}{1+(\sqrt{\mathcal{W}\mathcal{K}}F(\pi)/(k_s\pi))^2} > 1 \tag{27}$$

due to the interference.

To summarize, the interference of the wake force with the "head-tail force" of the kicker introduces an instability which is very similar to the head-tail instability. The strength of this instability is obtained by multiplying that of the head-tail instability by the factor, (27).

3.3 Interpretation of Each Term in Υ

In the above discussion, we find that the growth is due to the imaginary part of the parameter, Υ. Then, if the absolute value of imaginary part is much smaller than the real part, is the motion always stable? The answer is no. To understand what happens when the real part dominates, we examine two extreme cases. In both cases, we still assume that g is much smaller than unity.

The first case is that the kicker parameter, $\mathcal{K}$, is much larger than the wake parameter, $\mathcal{W}$. Υ is approximately given by $4\mathcal{K}/k_s$, which is a real quantity. But under the condition,

$$\Upsilon \simeq \frac{4\mathcal{K}}{k_s} > 2, \tag{28}$$

the eigenvalues of the transfer matrix, (25), become real, and the motion becomes unstable. The growth rate will be quite large, as in the case of the strong head-tail instability.

In the second case, when the second term of the real part in (23) dominates, a similar thing will happen. Under the condition,

$$|\Upsilon| = \left| -\frac{1}{a}\sqrt{\frac{\mathcal{W}}{\mathcal{K}}}\left(e^g - 1\right) \right| > 2$$

the eigenvalues of the transfer matrix become real. Remembering the definition of, g, (22), and considering the condition, $g \ll 1$, the above condition can be simplified to

$$\frac{\mathcal{W}}{k_s}\pi > 2.$$

This is nothing but the threshold behavior of the strong head-tail instability. Note that there is no enhancement in strength in this case, contrasted with the enhanced head-tail instability.

Strictly speaking, the analysis here is not perfectly rigorous, because we have started to solve the equation assuming that the solution is not a fast-varying function. But, by this analysis, we are able to understand the origin of each term in Υ.

3.4 Considerations with Concrete Numerical Values

In this section, we will give a numerical example, again using the KEKB parameters. For the transverse wake,[6] we will use the value, $W_0 \simeq 2 \times 10^{15} \Omega s^{-1} m^{-1}$. Other parameters used here are same as those used in the Section 2.6. From these values, we find (see expression (15) for the definition of the parameter, $\mathcal{W}$)

$$\mathcal{W} \approx 5 \times 10^{-6} m^{-1}. \tag{29}$$

Thus, with the KEKB parameters, we have an inequality,

$$\mathcal{K} \ll \mathcal{W} \ll k_s,$$

from the expressions, (13), (14) and (29). Then g is

$$g = \frac{\sqrt{\mathcal{K}\mathcal{W}}}{k_s} F(\pi) \approx \frac{5 \times 10^{-7}}{2 \times 10^{-5}} 2.4 = 0.04.$$

Since g is much smaller than unity, the enhancement factor, (27), is simply expressed by

$$f_e \approx 1 + g$$

and its numerical value is 1.04. One may think that this value might not introduce a serious situation. But in the case of KEKB, the growth rate of the head-tail instability itself can be strong if the chromaticity is chosen carelessly. Then this enhance factor can be critical in the operation.

Among the terms which form the parameter Υ, (see the expression (23)), the largest numerical contribution is given by the term

$$\frac{1}{a}\sqrt{\mathcal{W}/\mathcal{K}}(e^g - 1) \approx 0.15.$$

This is far less than 2, and no strong head-tail instability will occur.

In addition, the possible beam size-growth due to the kicker force alone is far from the threshold.
In fact, the parameter given by Equation (28) is only

$$\frac{4\mathcal{K}}{k_s} \approx 10^{-2} \ll 2.$$

4 DISCUSSION

So far we have discussed the effects of the feedback-kicker on the head-tail motion of a bunch. We have also taken its interference with a wake force into consideration. Based on these discussions, we have made numerical estimates of the effects for the KEKB machines. Here we give some general remarks on these effects. The key parameters in this discussion are $\mathcal{K}$, which is given by (6), and the enhancement factor, (27).

Looking at the definition of $\mathcal{K}$, we notice that effects of the kicker is stronger in smaller-size rings. Also we should note that a smaller beta-function at the kicker is preferable if we want to reduce the effects of the kicker. Usually, we desire a large beta-value at the kicker to minimize the voltage for a given damping rate. But we should reconsider this simple-minded rule of parameter choices in order to reduce the effects of the kicker.

The next thing to be pointed out is that a higher acceleration voltage (for a given value of the momentum compaction factor) is favorable from our point of view. The reason is simply that the bunch length, which appears in Equation (6), will be shorter, and the synchrotron frequency which appears in the denominator of the enhancement factor will be larger with the higher acceleration voltage.

5 CONCLUSIONS

We investigated the possible head-tail motion due to the head-tail force of the transverse feedback-kicker together with the short-range wake force. The analysis has been done with a simple two-particle model. If we take only the effects of the kicker into consideration, the transverse distance of the two particles, that can be understood as the transverse bunch size in an actual bunch, will grow with the growth rate of $\mathcal{K}\varphi c$. But the growth rate is expected to be much smaller than the damping rate due to synchrotron radiation for the design parameters of KEKB.

When we take also the wake force into account, the kicker force and the wake force interfere with each other. As a result, the usual head-tail instability is enhanced by the factor of $e^g/(1 + (g/\pi)^2)$. The numerical value of this enhancement factor is only 1.04 for the KEKB machine parameters. Although the enhancement factor is not very large, we should tune the chromaticity carefully during operation of the machine.

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References

[1] *KEKB B-Factory Design Report* (KEK 95–7, June 1995).

[2] *PEP-II An Asymmetric B-factory Conceptual Design Report* (LBL-PUB-5379, SLAC-418, CALT-68-1869, UCRL0ID-114055,UCIIPRA-93-01, June 1993).

[3] A matrix, A, that satisfies

$$A^tA = {}^tAA,$$

is called normal. Unitary matrices, Hermitian matrices and anti-Hermitian matrices satisfy the above condition. A normal matrix can be diagonalized under the similar transformation with some unitary matrix. But a matrix which does not satisfy the last condition is not diagonalizable with the unitary transformation.

[4] The two-particle model is clearly explained in A.W. Chao's text book, *Physics of Collective Beam Instabilities in High Energy Accelerators* (John Willey & Sons, Inc. 1993). The original papers concerning the two-particle model are cited in this book.

[5] There are two ways of defining the chromaticity, ξ. One definition gives the relation between energy deviation, δ, and the tune change ($\Delta\nu$) due to δ by

$$\Delta\nu = \nu\xi\delta,$$

where ν is the 0-current limit (non-perturbed) of the tune. On the other hand, with the other definition, the non-perturbed tune is taken in the chromaticity

$$\Delta\nu = \bar{\xi}\delta.$$

We adopted the first definition here.

[6] See section 5.2.2 of the reference.[1]